Published by Institute of Physics Publishing for SISSA

Received: *October 14, 2006*
Revised: *November 14, 2006*
Accepted: *December 11, 2006*
Published: *January 4, 2007*

The planar spectrum in $\mathrm{U}(N)$-invariant quantum mechanics by Fock space methods: I. The bosonic case

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Abstract: Prompted by recent results on Susy-$\mathrm{U}(N)$-invariant quantum mechanics in the large N limit by Veneziano and Wosiek, we have examined the planar spectrum in the full Hilbert space of $\mathrm{U}(N)$-invariant states built on the Fock vacuum by applying any $\mathrm{U}(N)$-invariant combinations of creation-operators. We present results about 1) the supersymmetric model in the bosonic sector, 2) the standard quartic Hamiltonian. This latter is useful to check our techniques against the exact result of Brezin et al. The SuSy case is where Fock space methods prove to be the most efficient: it turns out that the problem is separable and the exact planar spectrum can be expressed in terms of the single-trace spectrum. In the case of the anharmonic oscillator, on the other hand, the Fock space analysis is quite cumbersome due to the presence of large off-diagonal $O(N)$ terms coupling subspaces with different number of traces; these terms should be absorbed before taking the planar limit and recovering the known planar spectrum. We give analytical and numerical evidence that good qualitative information on the spectrum can be obtained this way.

Keywords: Matrix Models, 1/N Expansion, Field Theories in Lower Dimensions, Supersymmetric gauge theory.

JHEP01(2007)018

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1. Introduction

Interest in the planar (topological) expansion in QFT has always been alive since the work of 't Hooft [10] and Veneziano [12, 13] (see also [2]). A recent revival of interest has been triggered by a paper by Veneziano and Wosiek [15] (hereafter VW) followed by [16, 17], about a class of supersymmetric quantum mechanical models, the simplest among them being defined by

$$H = \{Q, Q^\dagger\},\ Q = \mathsf{Tr}(f^\dagger(a + g\,a^2)) \tag{1.1}$$

where f and a are $N \times N$ operator valued Hermitian matrices, with standard canonical (anti)commutation relations; the model reveals a rich structure in the spectrum, in particular a peculiar duality $b \to 1/b$, where $b^2 = g^2 N$ is 't Hooft's parameter. The approach to the planar limit is based on the Fock space representation of the Hamiltonian, which is for the time being the only technique available in cases where a change of variable to U(N)-invariant "radial" coordinates is not applicable [1, 6]. The results presented by VW are restricted to a peculiar subspace of the full sector of U(N) invariant states, namely the subspace $\mathcal{H}^{(0)}$ spanned by the vectors which are obtained by applying a "single trace operator" to the Fock vacuum

$$\mathsf{Tr}(a^{\dagger n})\,|0\rangle\,, \quad n = 0, 1, 2, \dots\,. \tag{1.2}$$

This subspace is considered "dominant", in the planar limit, with respect to other states built over the vacuum by applying any product of invariant operators. The Fock

space approach to the large N limit was considered in [11], where the issue of which kind of operators would leave the subspace of single-trace states invariant in the planar limit was addressed. The VW Hamiltonian (1.1) enjoys this property, while the anharmonic oscillator Hamiltonian

$$H = \frac{1}{2}\mathsf{Tr}(p^2 + q^2) + \frac{g^2}{N}\mathsf{Tr}(q^4) \tag{1.3}$$

does not. It follows that in this latter case it is necessary to analyze the Hamiltonian in the full Hilbert space of $\mathrm{U}(N)-$invariant states. In the SuSy case it is clear that restricting to $\mathcal{H}^{(0)}$ is legitimate in the planar limit, but one misses in this way a big portion of the spectrum, and, moreover, to estimate the $1/N$ corrections one has to study the full Hilbert space.

The multiplicity of states makes the calculation of the spectrum much harder. To be specific, at a given number of quanta, i.e. considering a product of n creation operators on the vacuum, there exists one state in $\mathcal{H}^{(0)}$ but there exist $p(n)$ states of the kind

$$\mathsf{Tr}(a^{\dagger\lambda_1})\,\mathsf{Tr}(a^{\dagger\lambda_2})\,\ldots\,\mathsf{Tr}(a^{\dagger\lambda_m})\,|0\rangle\,, \quad m \leq n \tag{1.4}$$

where λ is any partition of n and the function $p(n)$ counts them all. It is known since Hardy and Ramanujan that $p(n)$ grows exponentially ($p(n) = O(\exp\{K\sqrt{n}\})$, $K = \pi\sqrt{2/3}$), and this is the main obstacle in a purely numerical approach. For instance, while it is easy to diagonalize the VW Hamiltonian corresponding to a maximum number of quanta of thousands or even more, the mere fact that $p(200) \approx 4 \times 10^{12}$ (as it was first computed exactly by MacMahon [4]) puts a severe limit to the level one can explore numerically.

Our main result (section 2) will be the following: for the SuSy VW Hamiltonian in the bosonic sector the Hilbert space of $\mathrm{U}(N)$ invariant states splits into an infinite number of subspaces $\mathcal{H}^{(m)}, m = 1, 2, 3, \ldots$, which are left invariant by the Hamiltonian in the planar limit. These subspaces are characterized by states with a given number m of trace factors in eq. (1.4); so to speak the number of traces is a good planar quantum number. As a consequence, the Hamiltonian is recognized to be separable into the sum of commuting operators, all unitarily equivalent to the single trace operator studied by VW. Essentially the Hamiltonian is analogous to an isotropic multidimensional harmonic oscillator, where the one-dimensional operator is given by VW's operator. This fact has several consequences, for instance if the 't Hooft parameter $b^2 = g^2N$ is grater than one, then all levels are infinitely degenerate.

The corrections of $O(1/N)$ break this degeneracy since they couple the subspaces $H^{(m)}$ together. We do not have exact results on this breaking mechanism, only numerical evidence. It appears that while higher states' degeneracy is lifted, the first level (the zero mode) keeps its infinite degeneracy even after the $1/N$ corrections are taken into account.

For a second class of Hamiltonian operators, which typically do not leave the Fock vacuum invariant, as is the case for the anharmonic oscillator solved in [1], the analysis is more involved. We give evidence on how the exact result, derived by introducing "radial variables" for the N-dimensional matrix $\mathbf{x}$, can be recovered numerically by reabsorbing the off-diagonal $O(b^2N)$ terms which deny the possibility to have a straightforward planar limit. Our conclusion is that Fock space methods are not particularly convenient for this

kind of models, but they can still be useful to explore the spectrum numerically in those cases where a BIPZ technique is not applicable.

2. Susy quantum mechanics in the $\mathrm{U}(N)$ invariant sector

Let us introduce a bit of formalism. Let

$$|\lambda\rangle \equiv |\lambda_1, \lambda_2, \ldots, \lambda_m\rangle = \mathcal{N}_\lambda \, \mathsf{Tr}(a^{\dagger\lambda_1})\mathsf{Tr}(a^{\dagger\lambda_2}) \ldots \mathsf{Tr}(a^{\dagger\lambda_m})|0\rangle \, . \tag{2.1}$$

The right-hand side is symmetric in the exchange of λ's, hence the left-hand side will always be reduced to a normal form subject to the condition

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \, . \tag{2.2}$$

Where confusion may not arise, we shall denote the basis vectors simply by $|n, m, \lambda\rangle$ or simply $|\lambda\rangle$ where λ is an m-partition of $n \in \mathbb{Z}^+$, a non-negative integer; we write $|\lambda| = n$ or $\lambda \vdash n$ if λ is a partition of n as it is customary in the mathematical literature. The "one-trace-states" of ref. [15] are special cases, namely $|n, 1, \{n\}\rangle$.

If we refer our states to a basis of coherent states $|\mathbf{z}\rangle$, such that

$$a_{ij}|\mathbf{z}\rangle = \bar{z}_{ij}|\mathbf{z}\rangle \tag{2.3}$$

we realize that the states $\langle \mathbf{z}|n, m, \lambda\rangle$ realize a basis in the space of symmetric functions in the eigenvalues of $\mathbf{z}$. Other basis, such as Schur's, can also be used where convenient (see e.g. [5, 7]): in section 3 we shall turn to Schur's basis which coincides with the free eigenbasis (for a detailed description of this formalism, see [8]). One should take into account the fact that our basis is not orthogonal. This is not going to raise any problem, the only consequence being that the matrix representing the Hamiltonian is not symmetric, but still unitarily equivalent to a Hermitian one. It is however important to normalize the states at least to leading order, to avoid having unnatural dependence on N in the matrix elements. So we decide to fix the normalization factor to take into account the leading power in N:

$$\mathcal{N}_\lambda = N^{-|\lambda|/2} \, . \tag{2.4}$$

(this is just the scaling with N, the exact normalization is known, but not actually necessary, see appendix 2). Now we compute the matrix elements of the Hamiltonian introduced in eq. (1.1), as it is reduced to the bosonic sector. For details about the algebra involved the reader is referred to [14, 15]. We are led then to compute

$$\Big(\mathsf{Tr}(a^\dagger a) + g\mathsf{Tr}(a^{\dagger 2}a + a^\dagger a^2) + g^2\mathsf{Tr}(a^{\dagger 2}a^2)\Big) \; |\lambda\rangle = \sum_{\lambda'} \mathcal{H}_{\lambda',\lambda} \, |\lambda'\rangle \, . \tag{2.5}$$

By applying the commutation relations

$$[a_{ij}, a^\dagger_{kl}] = \delta_{il}\delta_{jk} \tag{2.6}$$

one must bring all annihilation operators to the right eventually reaching the vacuum. This is easily done symbolically using a language like Mathematica$^{(\mathrm{R})}$, but it is clear that the

operation is going to exhaust the capabilities of your computer as soon as the number $|\lambda|$ reaches ten or a little more, because the multiplicity of terms generated. A way out to this practical limitation is provided by the following analytic result (notice that the notation $\hat{\lambda}_j$ means that the term λ_j must be deleted from the list):

Theorem 1.

1. $$\mathsf{Tr}(a^{\dagger 2}\, a)\,|\lambda_1,\ldots,\lambda_m\rangle = N^{1/2}\sum_j \lambda_j\,|\lambda_1,\ldots,\lambda_j+1,\ldots,\lambda_m\rangle$$

2. $$\mathsf{Tr}(a^{\dagger 2}\, a^2)\,|\lambda_1,\ldots,\lambda_m\rangle = \sum_{\substack{1\le j\le m\\ \lambda_j>1}} \lambda_j \sum_{s=0}^{\lambda_j-2} |\lambda_1,\ldots,\lambda_j-s,\ldots,\lambda_m,s\rangle$$
$$+2\sum_{j<l}\lambda_j\,\lambda_l\,|\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\hat{\lambda}_l,\ldots,\lambda_m,\lambda_j+\lambda_l\rangle$$

3. $$\mathsf{Tr}(a^{\dagger}\, a^2)\,|\lambda_1,\ldots,\lambda_m\rangle = N^{-1/2}\sum_j \lambda_j \sum_{s=1}^{\lambda_j-1} |\lambda_1,\ldots,\lambda_j-s,\ldots,\lambda_m,s-1\rangle$$
$$+2\,N^{-1/2}\sum_{j<l}\lambda_j\,\lambda_l\,|\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\hat{\lambda}_l,\ldots,\lambda_m,\lambda_j+\lambda_l-1\rangle$$

Proof. The proof follows essentially the ideas introduced in [11]:

i. having just one annihilation operator to commute, the result is given by the sum of terms obtained by applying the commutator to each factor in $|\lambda\rangle$, and the result coincides with that one has for single trace states. The number of quanta on the right increases by 1, hence the power of N which takes into account the normalization. The resulting kets should be put in the standard form of eq. (2.2) after the operation has been completed.

ii. The two annihilation operators can be made to commute with the same factor $a^{\dagger\lambda_j}$, provided $\lambda_j > 1$, in which case we have a splitting into two traces or with two different factors, in which case we have a merge into one single trace. A factor 2 is due to the fact that the same pair (j,l) can be coupled in two different ways to a^2. The term with $s = 0$ produces a factor N and this is the dominant term in the result. As before the ket arguments on the right-hand-side should be rearranged in order to have a non-increasing m-tuple.

iii. As before, the two annihilation operators can be commuted with the same $\mathsf{Tr}(a^{\dagger\lambda_j})$ term, in which case we have a splitting into two traces; when $s = 1$ we get a factor N which gives the dominant contribution. Otherwise the two a's commute with two distinct factors and we have a merge into a single trace.□

Notice that in case (ii) the operator has a g^2 factor, which means that the dominant term $O(N)$ produces a finite result in the planar limit, the other being depressed by a factor $1/N$. In the other two cases the factor g combines with the $N^{1/2}$ factor to give again

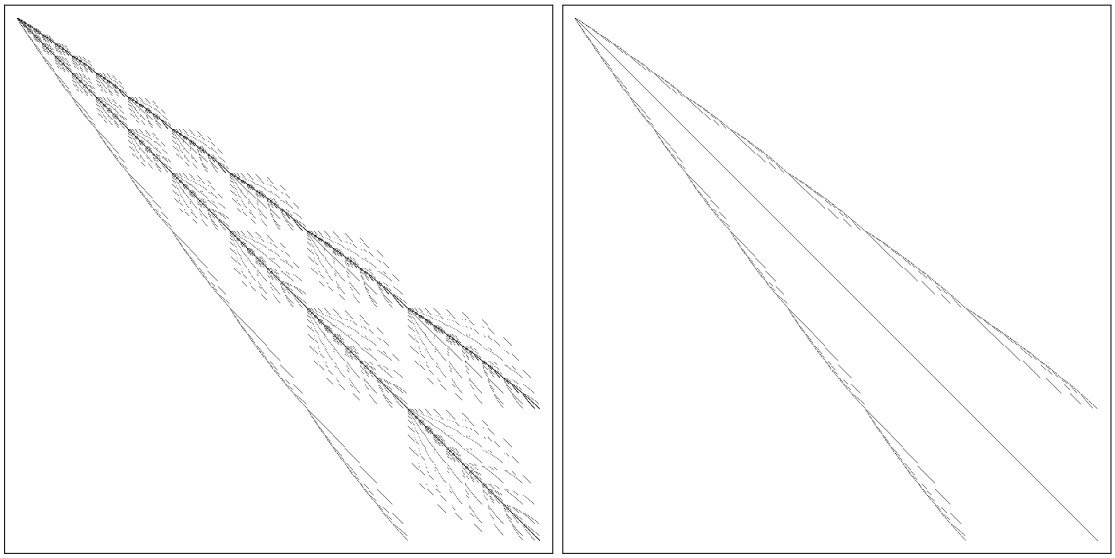

Figure 1: The sparsity pattern of $\mathcal{H}$ up to $\lambda \vdash 16$ (left); the same on the right after taking the planar limit.

a finite limit. Other terms provide off-diagonal corrections $O(1/N)$, which can be taken into account to get the planar expansion.

Finally we have the matrix elements independent from $g = 0$, that is $\mathsf{Tr}(a^\dagger\, a)|n, m, \lambda\rangle$; this is a diagonal operator, with eigenvalue $n = |\lambda|$, independently from the number of traces m.

To get an idea of the full matrix, consider the portrait in figure 1 (left), which includes all non-vanishing matrix elements, including the $O(1/N)$ terms, for $n \leq 16$ (914 states) Much simpler is the planar matrix, the finite part at $N = \infty$, which is depicted in figure 1 on the right. Both patterns are replicated with no substantial variations for higher values of n (we checked up to $n = 22$).

In spite of the apparent complexity of this matrix, which would discourage from pushing to higher partition numbers - remember the exponential growth of $p(n)$ - it turns out that the planar limit is under full *analytical* control. This is due to the following observation: the finite matrix elements in the planar limit are all generated when the Hamiltonian operator interacts with a single factor $\mathsf{Tr}(a^{\dagger\lambda_j})$, and the result is built by the sum of all these contributions. This means that in the planar limit the number of traces in $|\lambda\rangle$ is unchanged by the application of H, in other words the Hilbert space splits into invariant subspaces $\mathcal{H}^{(m)}$ characterized by a fixed number m of traces and the planar Hamiltonian is given by the sum of commuting operators H_j, each of them acting on the $j-th$ index in the partition. Moreover each H_j is identical to VW operator in the bosonic sector, that is the Hamiltonian restricted to single-trace states. Hence we have:

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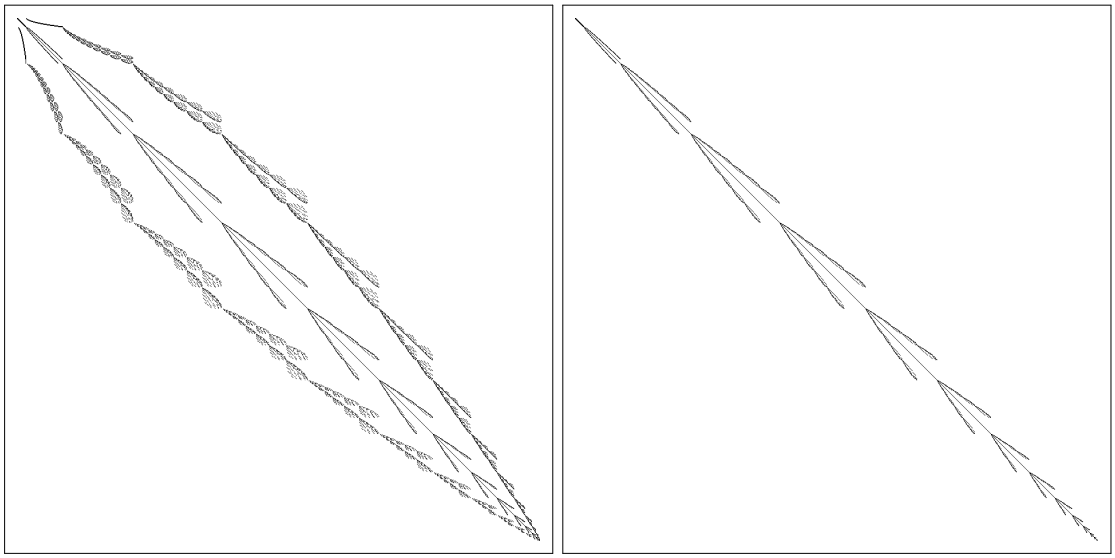

Figure 2: The rearrangement of $\mathcal{H}$ (left) and its block-diagonal structure in the planar limit (right).

Theorem 2. *The planar spectrum of the* VW *SuSy operator in the full Hilbert space of* $\mathrm{U}(N)$ *invariant states, as defined in eq. 2.5, is given by the union of the spectra in each subspace* $\mathcal{H}^{(m)}, m = 1, 2, 3, \ldots$, *namely*

$$E_{n_1,\ldots,n_m} = E^{(1)}_{n_1} + \ldots + E^{(1)}_{n_m},\ n_1 \geq n_2 \geq \ldots \geq n_m \tag{2.7}$$

where $E^{(1)}_n$ *are the eigenvalues of the Hamiltonian restricted to the single trace states as given in [15].*

Proof. The result follows from the property that the number of traces is left invariant in the planar limit; the constraint on the quantum numbers stems from the symmetry under permutation of the trace factors. (To make the paper more self contained, we give a derivation of the VW spectrum in appendix 1).□

Notice that the block-diagonal structure of the planar matrix can be made manifest if we rearrange the partitions in order of increasing number of traces, as shown in figure 2.

We tested the result by diagonalizing each diagonal block. In this case we can explore larger matrices and the result is checked to high accuracy.

Some comments on the structure of the spectrum are here in order. In the weak coupling limit the eigenvalues $E^{(1)}_n \approx (1+b^2)n, (n > 1)$, so the spectrum tends to be highly degenerate, like the spectrum of the union of m-dimensional harmonic oscillators for $m \geq 1$. Clearly, the states with m traces set in only starting with at least m quanta, which implies that the multiplicity of levels is not infinite, but it is steadily growing. See figure 3 for the case $b = 1/4$, where the typical $p(n)$ multiplicity is manifest.

The spectrum is dramatically different at strong coupling. As it has been shown in [15], there is a duality between weak and strong coupling, given by

$$(E^{(1)}_n(b) + 1)/b = b(E^{(1)}_{n+1}(1/b) + 1)\,,\ (b^2 = g^2 N < 1, n \geq 1)\,. \tag{2.8}$$

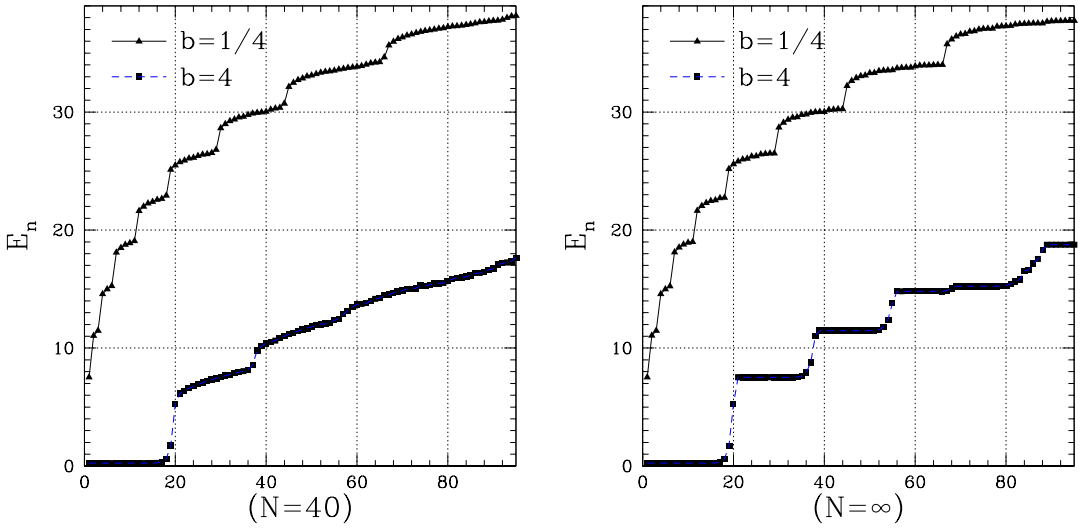

Figure 3: The spectrum of $\mathcal{H}$, truncated at $\lambda \vdash 22$, for $N = 40$ and $N = \infty$.

Moreover a new zero energy state arises at $b > 1$,$E_1^{(1)}(b) = 0$. Since the VW spectrum determines the whole planar spectrum, the duality relation extends, with slight modifications, to the general case. However the structure of the spectrum at strong coupling is deeply changed, namely *each eigenvalue is infinitely degenerate*. This is due to the fact that in the relation (2.7) we may choose $n_2 = n_3 = \ldots = n_m = 0$ corresponding to the zero modes and therefore there is an eigenvalue $E_n^{(1)}$ in each invariant subspace $\mathcal{H}^{(m)}$. This fact extends to all other eigenvalues: taking $n_3 = n_4 = \ldots = n_m = 0, m \geq 2$, one gets an infinite number of eigenstates with the same eigenvalue $E_{n_1}^{(1)} + E_{n_2}^{(1)}$ and so on.

This property can only be approximately displayed in a numerical plot, since the diagonalization process implies a truncation on the partition number, however a clear plateaus formation can be seen in figure 3 corresponding to $b > 1$ (the plateaus would be infinitely long in the exact solution).

What happens of this infinite degeneracy at finite N? We do not have analytic results on this matter. However diagonalizing the full Hamiltonian including all subleading terms one may hope to get some indications. In figure 3 we report the two cases $N = 40$ and $N = \infty$, truncated at $\lambda \vdash 22$ (a 4507×4507 sparse matrix). We may observe that the degeneracy of higher levels tends to be lifted by $O(1/N)$ corrections, while the ground state appears to be robust against the corrections. We are lead to conjecture that the zero modes are present even after switching on the subleading terms.

3. The $\mathrm{U}(N)$ symmetric anharmonic oscillator

The Hamiltonian of eq. 1.3 has been solved in the planar limit in [1, 6]. Here we want to analyze the problem from the point of view of Fock space. The Hamiltonian deeply differs from the SuSy model; for instance the Fock vacuum is not an eigenstate, secondly, and more important, the Hilbert subspaces $\mathcal{H}^{(m)}$ are *not* invariant. These facts make the

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Fock space approach much more involved than in the SuSy case. Nevertheless we find it instructing how the exact result may eventually emerge from the Fock space analysis.

The Hamiltonian contains three main contributions (see [14] where a similar Hamiltonian is studied):

$$\begin{aligned} H &= H_0 + H_2 + H_4 \\ H_0 &= \frac{1}{2}N^2 + \mathsf{Tr}(a^\dagger a) + \frac{b^2}{4N}\mathsf{Tr}(a^{\dagger 2}a^2 + \ldots + a^2 a^{\dagger 2}) \\ H_2 &= \frac{b^2}{4N}\mathsf{Tr}(a^{\dagger 3}\, a + \ldots + a^3\, a^\dagger) \\ H_4 &= \frac{b^2}{4N}\mathsf{Tr}(a^{\dagger 4} + a^4) \end{aligned} \tag{3.1}$$

where the dots point to all other products generated by $(a + a^\dagger)^4$, which differ from the terms displayed by the order of the operators. It would be nice to have simple recipes like in the previous section to compute matrix elements. In this case however the problem is much more involved, with 16 quartic operators, some of them containing two annihilation operators or more. To get an idea of the work required, just consider matrix elements between states with partition number (occupation number) ten, that is a 42×42 block on the diagonal: this makes 10584 matrix elements. If we have to reach a matrix truncation corresponding to a maximum $|\lambda|$ of order at least 20, the total number of matrix elements to be computed touches the 10^8 range, clearly unfeasible by brute force. But also an approach through a computer algebra system is going to face serious difficulties if we want to raise the truncation number above 10, because of the proliferation of terms in the reduction of operator products. So we need some kind of human-assisted-computer-algebra-approach.

3.1 Matrix elements by computer algebra at low $|\lambda|$

We have set up a Mathematica$^{(R)}$ code which is able to compute the matrix elements by blindly applying the commutation relations. Let us observe that since we initially work with the basis of eq. (2.1) which is not orthogonal, the matrix made of scalar products $\langle\lambda|\, H\, |\mu\rangle$ must be multiplied on the left by the inverse of the metric matrix $\mathcal{M}_{\lambda\mu} = \langle\lambda|\mu\rangle$ before we can use it to compute the spectrum. This is the true representative matrix that we denote by $\mathcal{H}$

$$\mathcal{H}_{\lambda\mu} = \mathcal{M}^{-1}_{\lambda\nu}\langle\nu|H|\mu\rangle \tag{3.2}$$

We start examining the main block on the diagonal of the Hamiltonian in the "partitions basis" $|\lambda\rangle$. From the analysis of low dimensional blocks a regular pattern emerges. After subtracting a term $\frac{1}{2}N^2(1+b^2)$ proportional to the identity operator, we are left with an "almost" diagonal matrix, in the sense that the off-diagonal matrix elements are depressed

by a factor $1/N$. For instance:

$$\mathcal{H}_{00} = \frac{1}{4}b^2$$

$$\mathcal{H}_{11} = \frac{1}{4}b^2 + 1 + 3b^2$$

$$\mathcal{H}_{\lambda\vdash 2,\mu\vdash 2} = \begin{pmatrix} \frac{1}{4}b^2 + 2(1+3b^2) & \frac{3b^2}{N} \\ \frac{3b^2}{N} & \frac{1}{4}b^2 + 2(1+3b^2) \end{pmatrix}$$

$$\mathcal{H}_{\lambda\vdash 3,\mu\vdash 3} = \begin{pmatrix} \frac{1}{4}b^2 + 3(1+3b^2) & \frac{6b^2}{N} & 0 \\ \frac{9b^2}{N} & \frac{1}{4}b^2 + 3(1+3b^2) & \frac{9b^2}{N} \\ 0 & \frac{3b^2}{N} & \frac{1}{4}b^2 + 3(1+3b^2) \end{pmatrix}$$

$$\mathcal{H}_{\lambda\vdash 4,\mu\vdash 4} = \begin{pmatrix} \frac{1}{4}b^2+4(1+3b^2) & \frac{9b^2}{N} & \frac{12b^2}{N} & 0 & 0 \\ \frac{12b^2}{N} & \frac{1}{4}b^2+4(1+3b^2) & 0 & \frac{12b^2}{N} & 0 \\ \frac{6b^2}{N} & 0 & \frac{1}{4}b^2+4(1+3b^2) & \frac{3b^2}{N} & 0 \\ 0 & \frac{9b^2}{N} & \frac{6b^2}{N} & \frac{1}{4}b^2+4(1+3b^2) & \frac{18b^2}{N} \\ 0 & 0 & 0 & \frac{3b^2}{N} & \frac{1}{4}b^2+4(1+3b^2) \end{pmatrix}$$

A part from the obvious occurrence of a multiple of the identity of dimension $p(n)$ with a value $n(1+b^2)+\frac{1}{4}b^2$, which obviously generalizes to all higher dimensional blocks, the rest of the matrix is not readily generalizable, at first sight.

The off-diagonal blocks are generated by H_2 and H_4. Let's examine the first few of them to see the pattern (we discard terms $O(1/N^2)$):

$$\mathcal{H}_{\lambda\vdash 2,\mu\vdash 0} = \begin{pmatrix} Nb^2 \\ \frac{1}{2}b^2 \end{pmatrix}$$

$$\mathcal{H}_{\lambda\vdash 3,\mu\vdash 1} = \begin{pmatrix} b^2 \\ Nb^2 \\ \frac{1}{2}b^2 \end{pmatrix}$$

$$\mathcal{H}_{\lambda\vdash 4,\mu\vdash 2} = \begin{pmatrix} 2b^2 & 0 \\ 0 & 2b^2 \\ Nb^2 & 0 \\ \frac{1}{2}b^2 & Nb^2 \\ 0 & \frac{1}{2}b^2 \end{pmatrix}$$

$$\mathcal{H}_{\lambda\vdash 0,\mu\vdash 2} = \begin{pmatrix} 2b^2N+\frac{b^2}{N} & 3b^2 \end{pmatrix}$$

$$\mathcal{H}_{\lambda\vdash 1,\mu\vdash 3} = \begin{pmatrix} 9b^2 & 2b^2N+13\frac{b^2}{N} & 9b^2 \end{pmatrix}$$

$$\mathcal{H}_{\lambda\vdash 2,\mu\vdash 4} = \begin{pmatrix} 12b^2 & 18\frac{b^2}{N} & 4b^2N+26\frac{b^2}{N} & 3b^2 & 0 \\ 12\frac{b^2}{N} & b^2 & 0 & 2b^2N+25\frac{b^2}{N} & 18b^2 \end{pmatrix}$$

The matrix elements for H_4 are similar in that they contain "superleading" terms $O(b^2N)$ which do not allow for a straightforward planar limit. Unlike the $O(N^2)$ along the diagonal which only contributes to the zero-point energy, the big off-diagonal terms cannot be disposed of in a simple way and they represent the main obstacle to compute the planar limit. With hindsight from [1], we know that the zero point energy has a full expansion in powers of b^2, which means that contributions coming from the off-diagonal terms are

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n	$\delta E = b^2/N \times$
0	0
1	0
2	(3,-3)
3	(9, 0,-9)
4	(18, 6, 0,-6,-18)
5	(30,15, 6, 0,-6,-15,-30)
6	(45,27,15, 9, 9, 0,-9,-9,-15,-27,-45)

Table 1: The fine structure of anharmonic oscillator to first order in $1/N$.

needed to recover the exact result. We shall postpone the discussion on this point after we switch to another basis, that of Schur's functions.

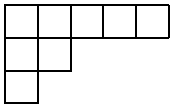
3.2 The basis of Schur's functions

It is well-known from the theory of symmetric functions (see e.g [5, 7]) that there are many interesting basis in the ring of symmetric functions on N indeterminates, $\Lambda(N)$. Up to now we worked with the "partitions basis" of eq. (2.1); it's easy to relate this basis to the corresponding basis in $\Lambda(N)$. Any vector λ can be represented in the basis of coherent states where it becomes a symmetric polynomial of the eigenvalues $\zeta_1, \ldots, \zeta_N$ of the matrix $\mathbf{z}$

$$\langle z|\lambda\rangle = \mathsf{Tr}(z^{\lambda_1}) \ldots \mathsf{Tr}(z^{\lambda_m}) = \sum \zeta_j^{\lambda_1} \sum \zeta_j^{\lambda_2} \ldots, \sum \zeta_j^{\lambda_m} . \tag{3.3}$$

It is now possible to consider other basis in Λ to explore the properties of H. One such basis is defined by Schur's functions (see [8] for an application). This task can be easily achieved by symbolic computer algebra. Let us denote by $\mathcal{H}^{\mathcal{S}}$ the representation of H in Schur's basis. The main feature of $\mathcal{H}^{\mathcal{S}}$ turns out to be the following: all blocks along the diagonal are now in *exact diagonal form*; the diagonal matrix elements in the $n-$th block are now easily identified, being given by the already computed $n(1+3b^2)+\frac{1}{4}b^2$ plus corrections $O(1/N)$ which follow a regular pattern, e.g. for $1 \leq n \leq 6$ they are given in table 1.

The pattern may not be clear at first sight, but it emerges immediately if we consider the partitions and Ferrers-Young diagrams associated to them, as it is familiar from the theory of the symmetric group; namely to each partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ it is useful to associate a diagram composed of m lines each containing λ_k square cells, i.e.

$\{5, 2, 1\} \rightarrow$ 

The formula reproducing the data of table 1 is extremely simple

$$\delta E_\lambda = 3 \sum_{(i,j)\in\lambda} (i-j)\, b^2/N. \tag{3.4}$$

where the pair (i, j) runs on the cells of the Ferrers-Young diagram of the partition (i is the column and j the row index, respectively). For instance at $n = 6$ we find

$$\lambda = \{6\} \quad \rightarrow \square\square\square\square\square\square \rightarrow 3 \times \sum \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \rightarrow 45$$

$$\lambda = \{5,1\} \rightarrow \begin{array}{l}\square\square\square\square\square \\ \square\end{array} \quad \rightarrow 3 \times \sum \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & & & & \end{bmatrix} \rightarrow 27$$

$\dots$

By construction the correction is opposite in sign for conjugate partitions, and it averages to zero on every diagonal block, a feature which was already noticed in [6].

Also off-diagonal blocks greatly simplify in Schur's basis. First of all, since the basis *is* orthogonal, $\mathcal{H}^{\mathcal{S}}$ is symmetric, hence we save half of the computing effort. Secondly, the superleading terms turn out to be very regular: all coefficients of $b^2 N$ are either 1 or -1, for instance

$$\mathcal{H}^{\mathcal{S}}_{\lambda\vdash 3,\mu\vdash 5} = b^2 N \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix} + \text{finiteterms}$$

3.3 The role of superleading terms

We now want to discuss how superleading terms of $O(b^2 N)$ contribute to the planar expansion and allow in principle to recover the known result. At this stage we have an expansion for the matrix representing the Hamiltonian in Schur's basis of the following form

$$\mathcal{H}^{\mathcal{S}} = \frac{1}{2} N^2 (1+b^2)\mathbb{I} + N\, T_1 + T_0 + N^{-1} T_{-1} + \dots \tag{3.5}$$

The term T_1 should be taken into account *before* taking the limit $N \to \infty$. It is clear that off-diagonal terms of order N will contribute corrections of order N^2 to the eigenvalues, however there should not be any higher order corrections, since there are no such terms in the planar expansion; moreover these corrections should be the same for all eigenvalues from what we know on the basis of the exact solution. The effect of T_1 should then reduce to a mere redefinition of the vacuum energy. We can deduce even more: T_1 cannot have by itself any discrete eigenvalue t_1, otherwise, in the strong coupling limit, we would find, by perturbation theory in $1/b$, a leading behavior $b^2 N t_1$ which is absent in the exact solution. Hence we are led to argue that T_1 must be an operator with continuous spectrum which perturbs T_0 by simply shifting the zero energy level, but leaving the energy gaps unchanged. An example of this kind comes from elementary quantum mechanics. Consider the simple harmonic oscillator perturbed by a term $-Fx$; its perturbative expansion stops at second order for all levels and its only effect is to shift all the spectrum by a fixed amount $-\frac{1}{2}F^2$. We argue that this is happening with T_1 and this can be proven by a direct analysis showing that T_1 is unitarily equivalent to a component of the position operator in an anisotropic two-dimensional harmonic oscillator (see appendix 2). As in the harmonic oscillator example, it would be natural to change the vacuum state in order to re-absorb the term T_1. This will not be attempted here and is left for future investigations.[1]

[1] This idea was formulated by G.Veneziano, private communication.

Let us observe that the general picture we get at this stage, a ground state proportional to $\epsilon(b)\,N^2$ and a splitting of order $1/N$ given by eq. (3.4) is qualitatively correct, and this result is due to the introduction of a convenient basis, provided by Schur's functions.

3.4 The Hamiltonian matrix from group theory

Even if Schur's basis allows for a better starting point in attacking the problem, still the limitations posed by the brute force symbolic calculation do not allow us to reach a reasonable truncation of the Hamiltonian. But there is another way in which the theory of symmetric functions, or in other words the theory of the symmetric group, can help in the calculation of matrix elements. Let us denote by $|\lambda\rangle_{\mathcal{S}}$ the vector corresponding to the Schur's function

$$s_\lambda(x) = \frac{\det(x_j^{\lambda_i+i-1})}{\det(x_j^{i-1})}$$

while we keep the symbol $|\lambda\rangle$ for the states constructed via application of multiple traces (eq. (2.1)). The transformation between the two basis

$$|\lambda\rangle_{\mathcal{S}} = \sum_\mu S_{\lambda\mu}|\mu\rangle \tag{3.6}$$

is well-known and is directly related to the character tables of S_n, the group of permutations (see [5], Ch. 4):

$$S_{\lambda\mu} = \frac{1}{n!} d_\mu\, \chi^\lambda_\mu$$

where $n = |\lambda| = |\mu|$, d_μ is the dimension of the irreducible representation indexed by μ and χ is its character. We show now how the knowledge of S can simplify the computation of the matrix elements. Let us compute

$$\begin{aligned}
\mathsf{Tr}(a^{\dagger 4})|\lambda\rangle_{\mathcal{S}} &= \sum_\mu S_{\lambda\mu}\mathsf{Tr}(a^{\dagger 4})|\mu\rangle \\
&= \sum_\mu S_{\lambda\mu}|\mu'\rangle\Big|_{\mu'=\mu\cup\{4\}} \\
&= \sum_{\mu,\nu} S_{\lambda\mu}S^{-1}_{\mu'\,\nu}|\nu\rangle_{\mathcal{S}}
\end{aligned}$$

(here $\lambda \cup \mu$ denotes the standard set-theoretic *union* of two lists). The point is that the character table can be easily built to rather large dimensions using the results of Jacobi and contemporaries and this allows to break the limit of the brute force calculation. The normalization factor is easily found to be

$${}_{\mathcal{S}}\langle\lambda|\mu\rangle_{\mathcal{S}} = \delta_{\lambda\mu} \prod_{(i,j)\in\lambda} (N+i-j)$$

where the indices i, j run on the cells of the Ferrers-Young diagram of the partition (i is the column and j the row index, respectively), e.g.

$$\{4,2,1\} \to \begin{array}{|c|c|c|c|}\hline 11 & 21 & 31 & 41 \\ \hline 12 & 22 \\ \cline{1-2} 13 \\ \cline{1-1}\end{array} \to \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & & \\ -2 & & & \end{bmatrix}$$

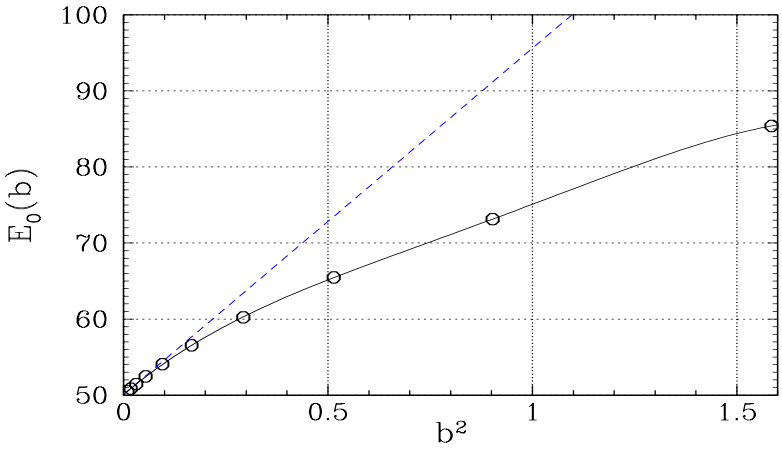


Figure 4: The ground state energy as a function of b^2 at $N = 10$.

which gives the normalization factor

$$_{\mathcal{S}}\langle\{4,2,1\}|\{4,2,1\}\rangle_{\mathcal{S}} = N(N+1)(N+2)(N+3)(N-1)N(N-2)$$

A similar formula can be found also for the matrix elements of H_2 in the Hamiltonian containing three creation and two annihilation operators. In this way we have built the matrix in the (orthonormal) Schur's basis up to $\lambda \vdash 21$ and presumably this may be improved. The matrix can be used to explore the spectrum in the planar limit by expanding in powers of $1/N$ and neglecting terms $O(1/N^2)$ or smaller. In particular we can study the crucial question about the ground state, which should be compared with the exact result (see figure 4 where the straight line corresponds to $\frac{1}{2}N^2(1+b^2)$). As a final comment, the introduction of Schur's basis appears the most natural choice in view of the structure of the exact solution, a collection of N non-interacting fermions (see [1]). Schur's functions are also known as Slater-Fock determinants, of course.

4. Conclusions

Fock space methods are very efficient for a class of U(N)-invariant Hamiltonians similar to VW model, where the Fock vacuum is stable. We derived the exact spectrum for the SuSy model in the bosonic sector, and this should be easily extended to the one-fermion sector. The one-trace sector, while representing a small subspace of the whole Hilbert space, provides the building block for understanding the whole spectrum. The $b^2 = 1$ transition marks a boundary between a typical approximate $p(n)$ degeneracy to a regime with infinite degeneracy, which is in general broken by $O(1/N)$ corrections which we have identified. Extension of these techniques to non-vanishing fermion number will be considered in a future publication. For the anharmonic oscillator, as well as for all other Hamiltonians covered by the exact BIPZ solution in the planar limit, Fock space methods are not convenient. Still, by a combined effort of analytical and computer algebra techniques, they may prove to be reliable in those cases where an approach based on "polar coordinates" à la BIPZ

is not available. Our results show that superleading matrix elements $O(b^2N)$ do not spoil the planar expansion, and the spectrum can be at least computed numerically.[2]

Acknowledgments

Several interesting discussions with G. Veneziano and J. Wosiek are gratefully acknowledged. We warmly thank G. Cicuta for his generous advice and fruitful collaboration while this work was conducted. We thank the Referee for many useful suggestions which helped in improving the final version of the paper.

A. The spectrum of the single-trace bosonic Hamiltonian

The spectrum of the SuSy model in the single trace subspace has been computed exactly in [15]. We give here an equivalent derivation, essentially algebraic in character.

The matrix to be diagonalized is the following

$$\langle n|\mathcal{H}|m\rangle = (1+b^2(1-\delta_{n1}))\, n\, \delta_{nm} + b\,\sqrt{m(m+1)}\delta_{n,m+1} + b\,\sqrt{n(n+1)}\delta_{m,n+1}\,.$$

Another matrix, differing only at the $(1,1)$ element turns out to be easily diagonalizable:

$$\langle n|\mathcal{H}_0|m\rangle = (1+b^2)\, n\, \delta_{nm} + b\,\sqrt{m(m+1)}\delta_{n,m+1} + b\,\sqrt{n(n+1)}\delta_{m,n+1}\,.$$

To see this, we first change to another representation

$$\psi_n = (-)^n b^n \sqrt{n}\phi_n$$

its action being given by

$$\mathcal{H}_0\phi_n = (1+b^2)\, n\phi_n - b^2(n+1)\phi_{n+1} - (n-1)\phi_{n-1} \tag{A.1}$$

Lemma 1. *$\mathcal{H}_0$ leaves all the subspaces $\mathcal{P}^{(k)} = \left\{\phi_n = \mathcal{N} n^k + \mathcal{O}(n^{k-1})\right\}$, $(k = 0, 1, 2, \ldots)$, invariant.*

Proof. By inspection, the terms n^{k+1} exactly cancel and the n^k term is multiplied by $(1-b^2)(k+1)$.□

Theorem 3. *$\mathcal{H}_0$ has discrete spectrum given by $E_n = |1-b^2|\, n, (n = 1, 2, 3, \ldots)$.*

*Proof.*In each finite-dimensional subspace $\mathcal{P}^{(k)}$ the matrix is upper-triangular, hence its eigenvalues can be read off the diagonal. The eigenvectors are given by the basis of orthogonal polynomials w.r.t. the discrete measure

$$|\phi|^2 = \sum_{n=1}^{\infty} n\, b^{2n}\, |\phi_n|^2$$

[2] Mathematica$^{(R)}$ codes used in this work will be made available to interested readers, on request.

which are known as Meixner polynomials [9]. They can be defined by

$$\phi_n^{(k)} = (1-b^2)\sqrt{\frac{k+1}{b^{2(k+1)}}}\,{}_2F_1(-k, n+1, 2, 1-b^2)$$

and satisfy the orthonormality property

$$\sum_{n=1}^{\infty} n\, b^{2n} \phi_n^{(k)} \phi_n^{(h)} = \delta_{k,h} \qquad \square$$

The rank-one-perturbation property [3] gives now the spectrum of $\mathcal{M}$ in the $F = 0$ sector. The eigenvalue equation reads

$$1 = b^2 \sum_{k=0}^{\infty} \frac{|\langle v_1 | \phi^{(k)} \rangle|^2}{(k+1)(1-b^2) - E}$$

where v_1 is the first basis vector in the representation of (1.1), and we find

$$b^2 \sum_{k=1}^{\infty} \frac{k b^{2(k-1)} (1-b^2)^2}{(1-b^2)\, k - E} = 1$$

hence, by setting $E = (1-b^2)\epsilon$, one gets

$$\Delta(\epsilon) = \sum_{k=1}^{\infty} \frac{k\, b^{2k}}{\epsilon - k} = \frac{1}{1-b^2}$$

The equation can be re-expressed in terms of hypergeometric functions:

$$\Delta(\epsilon) = 1 + \frac{\epsilon b^2}{\epsilon - 1}\, {}_2F_1(1, 1-\epsilon, 2-\epsilon, b^2)$$

which is equivalent to what has been derived in [15] by another method. In our derivation the structure of $\Delta(\epsilon)$ arises naturally via the "rank-one-perturbation" theorem.

B. Analysis of superleading terms

We want to discuss the structure of the superleading term in eq. (3.5). Since we work at $N = \infty$ we can study the representation of T_1 in the partitions basis, which in this limit is orthogonal. The leading behavior of the normalization factor can be easily expressed by representing a partition in terms of "composition", i.e.

$$\lambda = \{1^{r_1}\, 2^{r_2}, \ldots k^{r_k}\}$$

the integers r_j denoting the multiplicity of j in the partition of $n = \sum_j j\, r_j$. We find

$$\mathcal{N}_\lambda^{-2} = \langle \lambda | \lambda \rangle \approx N^{|\lambda|} \prod_j j^{r_j}\, r_j!$$

Now let's apply $\mathsf{Tr}(a^{\dagger 4})$ to the normalized $|\lambda\rangle$. We find

$$\mathsf{Tr}(a^{\dagger 4})|\lambda\rangle = \mathsf{Tr}(a^{\dagger 4})\,|r_1, r_2, \ldots, r_k\rangle \approx \frac{\mathcal{N}_\lambda}{\mathcal{N}_{\lambda'}}|r_1, r_2, \ldots, r_4+1, \ldots\rangle = 2N^2\sqrt{r_4+1}\,|\lambda'\rangle$$

where $\lambda' = \lambda \cup \{4\}$. The operator $\mathsf{Tr}(a^{\dagger 4})$ when multiplied by $\frac{1}{4}b^2/N$ gives rise to one component of T_1 which couples states in the form

$$\lambda \to \lambda \cup \{4\} \to \lambda \cup \{4^2\} \to \lambda \cup \{4^n\} \ldots$$

with matrix elements which are identical to those of a single creation operator. The action of $\mathsf{Tr}(a^4)$ is simply obtained by transposition; if a partition does not contain 2 or 4 it is annihilated by $\mathsf{Tr}(a^4)$, which means, of course, that the resulting matrix elements are not $O(N)$, but finite. The analysis of the remaining operators in T_1, coming from H_2, is much more involved, but the structure can be identified by using computer algebra. Eventually we get the following representation

$$\begin{aligned} T_1|r_1, r_2, \ldots, r_4, \ldots\rangle = {} & \sqrt{2}b^2 N(\sqrt{r_2}\,|r_1, r_2-1, \ldots, r_4, \ldots\rangle + \sqrt{r_2+1}|r_1, r_2+1, \ldots, r_4, \ldots\rangle) \\ & + \frac{1}{2}\,b^2 N(\sqrt{r_4}|r_1, r_2, \ldots, r_4-1, \ldots\rangle + \sqrt{r_4+1}\,|r_1, r_2, \ldots, r_4+1, \ldots\rangle) \end{aligned}$$

Notice that in this way we can identify an infinite sequence of subspaces invariant under the action of T_1: each subspace is labeled by a partition characterized by $r_2 = r_4 = 0$. This is a "vacuum" state for T_1. The action of T_1 builds a subspace isomorphic to the Hilbert space of a two-dimensional harmonic oscillator and the action of T_1 in all subspaces is always the same. The harmonic oscillator frequencies can be read off the diagonal part T_0, namely $(\omega_1, \omega_2) = (2, 4)(1 + b^2)$. Hence we find that indeed the only effect of T_1 at this level consists in a shift of the whole spectrum proportional to $b^2\,N$ exactly as it happens to a simple harmonic oscillator under perturbation by a term $\propto (a + a^\dagger)$.

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